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GENERALIZED HUMBERT CURVES*

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ABSTRACT

In this note we consider a certain class of closed Riemann surfaces which are a natural generalization of the so called classical Humbert curves. They are given by closed Riemann surfaces S admitting $H \cong \mathbb{Z}_2^k$ as a group of conformal automorphisms so that S/H is an orbifold of signature $(0, k+1; 2, \ldots, 2)$. The classical ones are given by $k = 4$. Mainly, we describe some of its generalities and provide Fuchsian, algebraic and Schottky descriptions.

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1. Introduction

In 1894, while investigating a net of conics, G. Humbert [12] encountered a family of curves $C_7 \subset \mathbb{P}^3(\mathbb{C})$ of genus $g = 5$. Later, in 1907, the same curves were encountered by Baker [3], related to a Weddle surface. If we denote by $\omega_1, \ldots, \omega_5$ five fundamental points of $\mathbb{P}^3(\mathbb{C})$, as a curve C_7 lies in each cone with vertex at ω_i over an elliptic curve E_i, there are five branched coverings $\pi_i : C_7 \to E_i$ and C_7 possesses five everywhere finite elliptic integrals. Moreover, each curve C_7 is invariant for each reflection h_i of $\mathbb{C}^4$ centered at the line corresponding to ω_i, and then C_7 is invariant under a group H of birational transformations isomorphic to $\mathbb{Z}_2^4$. We quote Humbert [12, pp. 140]: "La courbe C_7 et, par suite, les courbes σ_6 et $\tilde{\omega_6}$ donnent un exemple de courbes de genre cinq ayant cinq intégrales de première espèce réductibles aux intégrales elliptiques; de plus ces courbes possèdent cinq transformations birrationnelles en elles-mêmes. **Il serait intéressant de rechercher, d'une manière général, les courbes algébriques possédant des propiétés analogues**." In this way, a curve C_7 is a closed Riemann surface of genus $g = 5$ admitting a group $H \cong \mathbb{Z}_2^4$ of conformal automorphisms so that S/H is the Riemann sphere marked at exactly 5 points of order two. In this article we will study compact Riemann surfaces (algebraic curves) that possess properties analogous to those of Humbert curves C_7. Some facts about Humbert curves, mainly from the point of view of algebraic geometry, may be found, for instance, in [1, 5, 12, 18].

An orbifold $\mathcal{O}$ consists of a Riemann surface S (the underlying Riemann surface structure of $\mathcal{O}$) and a finite set (possibly empty) of conical points p_j in S of order n_j. The signature of the orbifold $\mathcal{O}$ is $(\gamma, k; n_1, \ldots, n_k)$, where γ denotes the genus of S. An automorphism of the orbifold $\mathcal{O}$ is a conformal automorphism of the underlying Riemann surface structure which permutes the conical points preserving orders. The group of automorphisms of the orbifold $\mathcal{O}$ will be denoted by $\mathrm{Aut}_{\mathrm{orb}}(\mathcal{O})$. We use the symbols "$\leq$"and "$\trianglelefteq$" to denote "subgroup" and "normal subgroup", respectively.

A closed Riemann surface S of genus g is called a **generalized Humbert curve of type** k, where $k \geq 1$, if it has a conformal group of automorphisms $H \cong \mathbb{Z}_2^k$ (direct sum of k copies of $\mathbb{Z}_2$) such that S/H has signature $(0, k+1; 2, \ldots, 2)$ (that is, the Riemann sphere with $k+1$ conical points of order 2); the group H will be called a **generalized Humbert group of type** k. Observe that the case $k = 4$ corresponds exactly to the case $S = C_7$. A pair (S, H), where S is a generalized Humbert curve of type k and $H \leq \mathrm{Aut}(S)$ is a generalized Humbert group of type k, is called a **generalized Humbert pair of type** k. If (S, H) is a generalized Humbert pair of type k, then the Riemann-Hurwitz formula asserts that the genus of S is $g_k = 1 + 2^{k-2}(k-3)$. In particular, for $k \geq 3$, the type k is uniquely determined by the genus g_k of S. Two generalized Humbert pairs (S_1, H_1) and (S_2, H_2) are called **conformally equivalent** if there exists a conformal homeomorphism $\varphi : S_1 \to S_2$ such that $\varphi^{-1} H_2 \varphi = H_1$. The locus of conformal classes of generalized Humbert curves of genus g_k, in the corresponding moduli space, has complex dimension $(k-2)$ for $k \geq 3$ and 0 for $k = 2$. We will see in Remark 2.9 that this locus is connected.

A **generalized Humbert curve** is a closed Riemann surface S admitting an Abelian group $H \leq \mathrm{Aut}(S)$ of conformal automorphisms so that S/H is the Riemann sphere and S is the homological covering of S/H. In the case that S/H is of signature $(0, k+1; 2, \ldots, 2)$ we obtain the generalized Humbert curves of type k. If the signature of S/H is $(0, k+1; n, \ldots, n)$, then $H \cong \mathbb{Z}_n^k$ and we obtain the generalized Fermat curves [7]. The most general situation will be discussed elsewhere. A higher dimensional situation may be found in [10].

1.1. Spherical Humbert curves. The only spherical orbifolds of signature $(0, k+1; 2, \ldots, 2)$ are given for $k = 1, 2$ as follows.

(1) $k = 1$ corresponds to $S = \widehat{\mathbb{C}}$ with $H = \langle J(z) = -z \rangle$;
(2) $k = 2$ corresponds to $S = \widehat{\mathbb{C}}$ with $H = \langle J(z) = -z, T(z) = 1/z \rangle$.

In particular, the Riemann sphere is a generalized Humbert curve of both types $k = 1, 2$. We see that for $g = 0$ (that is, for the Riemann sphere) there are generalized Humbert groups of two different types, contrary to the case $g \geq 1$.

1.2. Euclidean Humbert curves. The only Euclidean orbifolds of signature $(0, k+1; 2, \ldots, 2)$ are given by $k = 3$. To each complex number τ with positive imaginary part we can associate the torus $S = \mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau = \langle A(z) = z + 1, B_\tau(z) = z + \tau \rangle$. Furthermore, observe that $H = K_\tau/\Lambda_\tau$, where $K_\tau = \langle \Lambda_\tau, J(z) = -z, L(z) = -z + 1/2, M(z) = -z + 1/2 + \tau/2 \rangle$ acts on S with quotient orbifold of signature $(0, 4; 2, 2, 2, 2)$. In particular, every genus one Riemann surface is a generalized Humbert curve of type $k = 3$.

1.3. Hyperbolic Humbert curves. If $k \geq 4$, then every orbifold of signature $(0, k+1; 2, \ldots, 2)$ is hyperbolic. We will mainly deal with this case in the rest of this work. The first three hyperbolic ones are given by the pairs $(k, g_k) \in \{(4,5),(5,17),(6,49)\}$.

1.4. Standard Generators. Let (S,H) be a generalized Humbert pair of type k. It is clear from the signature $(0,k+1;2,\ldots,2)$ of S/H that there are precisely $k+1$ involutions in H with fixed points; in fact, any k of them can be chosen as generators of H, and the remaining one is given by the product of the chosen k. Such a set of generators of H will be called a **standard set** of generators, and any of the involutions with fixed points will be called a **standard generator** of H.

1.5. Towers of generalized pairs.

1.5.1. *Going down.* Let (S,H) be a generalized Humbert pair of type $k \geq 2$ and h be a standard generator of H. The cardinality of the set of fixed points of h is clearly 2^{k-1}, and, correspondingly, there is a two-fold branched covering

$$S \longrightarrow S/\langle h\rangle = \mathcal{O}$$

where the quotient orbifold $\mathcal{O}$ has signature $(1+2^{k-3}(k-4), 2^{k-1}; 2,\ldots,2)$, whose underlying Riemann surface structure is naturally a generalized Humbert curve of type $k-1$, with generalized Humbert group induced by $H/\langle h\rangle$.

1.5.2. *Going up.* Conversely, let us consider a generalized Humbert pair (S,H) of type $k \geq 1$ and let $h_1,\ldots,h_k$, $t = h_1h_2\cdots h_k$ the standard generators of H. Consider a closed disc $D \subset S$, containing exactly one fixed point of t, so that $t(D) = D$ and $h(D)\cap D = \emptyset$, for all $h \in H - \langle t\rangle$. Choose a point $p \in D$ so that $t(p) \neq p$ and a simple arc $\alpha \subset D$ connecting p with $t(p)$ so that $t(\alpha) = \alpha$ (in particular, α contains a fixed point of t). Now consider the two-fold branched cover of S defined by the following properties: (i) any simple loop on $S - \bigcup_{h\in H} h(D)$ lifts to exactly 2 simple loops and (ii) the branching locus is exactly at the the points $h(p)$, for $h \in H$. This is given by cutting S along the arcs $h(\alpha)$, for $h \in H - \langle t\rangle$, and gluing two copies of S (similar to the construction of hyperelliptic Riemann surfaces from the Riemann sphere). In this way we obtain a closed Riemann surface $\widehat{S}$ and a two-fold branched cover $\pi : \widehat{S} \to S$; this cover corresponds to a conformal involution $u : \widehat{S} \to \widehat{S}$. Every $h \in H$ lifts to a conformal involution $\widehat{h}$ so that $\widehat{h}u = u\widehat{h}$ and $\widehat{H} = \langle \widehat{h_1},\ldots,\widehat{h_k}, u\rangle \cong \mathbb{Z}_2^{k+1}$. It is not difficult to see that $(\widehat{S},\widehat{H})$ is a generalized Humbert pair of type $k+1$ so that $\widehat{h_1},\ldots,\widehat{h_k}$, u and $\widehat{h_1}\cdots\widehat{h_k}u$ are its standard generators. This construction may also be described as follows. Consider a generalized Humbert pair (S, H) of type k, and on the orbifold S/H choose any point x_1 which is disjoint from the conical set. Now consider the new orbifold, say $\mathcal{O}$, given by adding the point x_1 as a conical point of order 2 to the previous ones on S/H. Let Γ be the orbifold fundamental group of $\mathcal{O}$. Then, the commutator subgroup Γ' provides a generalized Humbert pair $(\widetilde{S}, \widetilde{H})$, with the property that on Γ there is an element of order 2, say w, so that if we denote by N the smallest normal subgroup of Γ containing w, then $K = \Gamma/N$ provides a uniformization of S/H and K' provides an uniformization of S.

This work is organized as follows. In Section 2 we prove and pairs Humbert generalized hyperbolic of case the consider their non-hyperellipticity and their non-trigonality. We also provide Fuchsian uniformizations and we observe the topological rigidity of a generalized Humbert action for any fixed type k. In Section 3 we prove that two generalized Humbert groups acting on the same generalized Humbert curve S are conjugate in the full automorphism group of S. In Section 4 we provide an algebraic description of the conformal classes of generalized Humbert pairs, that allows us to characterize their moduli spaces. In Section 5 we give a Schottky uniformization for generalized Humbert pairs; we also study their uniformizations by some special groups, the generalized Humbert–Whittaker groups. We thank the referee for sugestions that improved the presentation.

2. Hyperbolic generalized Humbert curves

From this section on we restrict to the case of hyperbolic generalized Humbert curves; that is, $k \geq 4$ or, equivalently, $g \geq 5$. Here we prove three facts about them: (1) non-hyperellipticity, (2) topological rigidity, and (3) non-trigonality. We also give their Fuchsian uniformizations in terms of commutator subgroups.

2.1. Non-Hyperellipticity of hyperbolic generalized Humbert curves. Classical Humbert curves C_7 (see the Introduction) are known to be non-hyperelliptic Riemann surfaces of genus 5. We proceed to see that this property also holds for the generalized Humbert curves.

Theorem 2.1: *Hyperbolic generalized Humbert curves are non-hyperelliptic.*

Proof. Assume we have a generalized Humbert pair (S, H) of type $k \geq 4$ and let us suppose S is hyperelliptic. In this case we have a branched two-fold holomorphic covering $Q : S \to \widehat{\mathbb{C}}$. As the hyperelliptic involution is in the center of the group of conformal automorphisms of S, H induces a group of Möbius transformations isomorphic to either $\mathbb{Z}_2^k$ or $\mathbb{Z}_2^{k-1}$. As each of any two different commuting Möbius transformations of order two permutes the fixed points of the other [15], the only possible groups are (i) the trivial group, or (ii) $\mathbb{Z}_2$, or (iii) $\mathbb{Z}_2^2$; in particular, $k \in \{1, 2, 3\}$, a contradiction. ∎

2.2. Non-trigonality of hyperbolic generalized Humbert curves. It is a well-known fact that each Humbert curve of genus $g = 5$ is non-trigonal [18]; that is, it has no holomorphic map of degree 3 onto the Riemann sphere. We now prove that this holds for all generalized Humbert curves of type k with $k \geq 4$. In fact this is a consequence of [14]; but for the sake of completeness we now give a short proof, which is just a simple modification of an argument due to R. Accola in [1].

Theorem 2.2: *Hyperbolic generalized Humbert curves are non-trigonal.*

Proof. Let S be a hyperbolic generalized Humbert curve. As we saw in Section 1.4, there is a branched covering $\pi : S \to R$, of some finite degree d, where R is a Humbert curve of genus 5. Let us assume S is trigonal, and consider a degree three branched covering $f : S \to \widehat{\mathbb{C}}$. For each $x \in R$ we consider $\pi^{-1}(x) = \{y_1, \ldots, y_r\}$. Let n_j be the degree of π at y_j, for $j = 1, \ldots, r$. Set $g : R \to \widehat{\mathbb{C}}$ by the rule $g(x) = f(y_1)^{n_1} \cdots f(y_r)^{n_r}$. Then g has degree 3, a contradiction to the non-trigonality of R. ∎

2.3. Fuchsian uniformizations. Since every hyperbolic Riemann surface may be uniformized by a Fuchsian group, we now provide a Fuchsian uniformization for hyperbolic generalized Humbert pairs by the commutator subgroup of appropriate genus zero Fuchsian groups.

Theorem 2.3: *Let (S, H) be a hyperbolic generalized Humbert pair. If Γ denotes a Fuchsian group (acting on the unit disc Δ) uniformizing the orbifold S/H, then (S, H) is conformally equivalent to $(\Delta/\Gamma', \Gamma/\Gamma')$, where Γ' denotes the commutator subgroup of Γ.*

Proof. Since $k \geq 4$, the quotient orbifold S/H of signature $(0, k+1; 2, \dots, 2)$ admits a uniformization by a Fuchsian group with presentation as follows.

$$\Gamma = \langle x_1, \dots, x_{k+1} : x_1^2, \dots, x_{k+1}^2, x_1 x_2 \cdots x_{k+1} \rangle. \tag{2.4}$$

The surface S is then uniformized by a torsion free normal subgroup F of Γ so that

$$\Gamma/F \cong H.$$

As F is torsion free, we have that $x_j \notin F$, for $j = 1, \dots, k+1$, and as $H \cong \mathbb{Z}_2^k$, we also have that $(x_i x_j)^2 \in F$, for each $i, j \in \{1, \dots, k+1\}$. Let us consider the normal closure, say Γ_2, in Γ of the collection

$$\{(x_i x_j)^2 : 1 \leq i, j \leq k+1\}.$$

Then $\Gamma_2 = \Gamma'$ and $\Gamma' \leq F \leq \Gamma$. Since $\Gamma/\Gamma' \cong H$, we obtain $\Gamma_2 = \Gamma' = F$. ∎

Corollary 2.5: *Let (S, H) be a hyperbolic generalized Humbert pair. Then the normalizer of H in $\mathrm{Aut}(S)$, denoted by $\mathrm{Aut}_H(S)$, is obtained as the lifting of $\mathrm{Aut}_{\mathrm{orb}}(S/H)$ under the natural regular branched covering $S \to S/H$. In particular, $\mathrm{Aut}_H(S)/H$ is isomorphic to $\mathrm{Aut}_{\mathrm{orb}}(S/H)$.*

Later, in Section 3, we will obtain that for $k \in \{4, 5\}$ Humbert groups are necessarily unique in the corresponding Humbert curves. In these cases, Corollary 2.5 tells us that $\mathrm{Aut}_H(S) = \mathrm{Aut}(S)$; in particular, this provides a simple way to compute the possible total automorphism groups for Humbert curves.

For every $k \geq 4$, let (S, H) be a generalized Humbert pair of type k. As $\mathrm{Aut}_{\mathrm{orb}}(S/H)$ is a finite group of Möbius transformations, and these are all known, one may explicitly compute $\mathrm{Aut}_H(S)$. In the following result we present some possibilities that appear for all possible values of k.

Proposition 2.6: *For every $k \geq 4$, there are three generalized Humbert curves of type k admitting larger groups of automorphisms with triangular signature. For each even (respectively odd) $k \geq 4$ there is one (respectively two) family(ies) of generalized Humbert pairs of type k admitting a group of conformal automorphisms isomorphic to a $\mathbb{Z}_2$-extension of the generalized Humbert group. The corresponding signature σ and uniformizing group G_r for the quotient orbifolds are given as follows, where the subindex r for the group indicates its order.*

(1) $\sigma = (0, 3; k+1, 4, 2)$, $r = 2^{k+1} \cdot (k+1)$ *and* $G_{2^{k+1} \cdot (k+1)}$ *has presentation*

$$\langle x, y : x^{k+1}, y^4, (xy)^2, (x^j y)^4, 3 \leq j \leq 2r \rangle \cong D_{k+1} \ltimes H.$$

(2) $\sigma = (0,3;2k,k,2)$, $r = 2^k \cdot k$ *and* $G_{2^k \cdot k}$ *has presentation*

$$\langle x, y : x^{2k}, y^k, (xy)^2, (x^{k+1}y)^2, (x^j y^j)^2, 1 \leq j \leq r+1 \rangle \cong \mathbb{Z}_k \ltimes H.$$

(3) $\sigma = (0,3;2(k-1),4,2)$, $r = 2^{k+1} \cdot (k-1)$ *and* $G_{2^{k+1} \cdot (k-1)}$ *has presentation*

$$\langle x, y : x^{2(k-1)}, y^4, (xy)^2, (x^{k-1}y^2)^2, (x^{2k-3}y^3)^2, (x^j y)^4, 3 \leq j \leq r+1 \rangle \cong D_{k-1} \ltimes H.$$

(4) *For k even, let $t = k/2$. Then $\sigma = (0, t+2; 4, 2, 2, \ldots, 2)$, from where the family is $(t-1)$-dimensional, $r = 2^{k+1}$, and the group $G_{2^{k+1}}$ has presentation*

$$\langle x, y, z_1, \ldots, z_t : x^4, y^2, z_j^2, xyz_1 \ldots z_t, (z_i z_j)^2, (x^2 z_j)^2, (x^3 z_j x^3)^2, (x^3 z_j x z_i)^2,\ 1 \leq i,\ j \leq t \rangle \cong \mathbb{Z}_2 \ltimes H.$$

(5) *For k odd, let $t = (k+1)/2$. Then $\sigma = (0, t+2; 2, 2, 2, \ldots, 2)$, from where the family is $(t-1)-$dimensional, $r = 2^{k+1}$, and $G_{2^{k+1}}$ has presentation*

$$\langle x, y, z_1, \ldots, z_t : x^2, y^2, z_j^2, xyz_1 \ldots z_t, (z_i z_j)^2, (z_i x z_j x)^2,\ 1 \leq i,\ j \leq t \rangle \cong \mathbb{Z}_2 \ltimes H.$$

(6) *For k odd, let $t = (k-1)/2$. Then $\sigma = (0, t+2; 4, 4, 2, \ldots, 2)$, from where the family is $(t-1)$-dimensional, $r = 2^{k+1}$, and $G_{2^{k+1}}$ has presentation*

$$\langle x, y, z_1, \ldots, z_t : x^4, y^4, z_j^2, xyz_1 \ldots z_t, (z_i z_j)^2, (z_j x^2)^2, (x z_j x)^2, (z_i x z_j x^3)^2,\ 1 \leq i,\ j \leq t \rangle \cong \mathbb{Z}_2 \ltimes H.$$

Remark 2.7: Note that the curves in the first three cases of Proposition 2.6 correspond to considering the groups acting on the orbifold S/H given by the dihedral group D_{k+1} of order $2(k+1)$, the cyclic group of order k, and the dihedral group D_{k-1} of order $2(k-1)$ respectively. Therefore, for large values of k, these are precisely the curves with largest group of automorphisms $\mathrm{Aut}_{\mathrm{orb}}(S/H)$ in the family of generalized Humbert curves of type k. In the case $k = 4$, the first four cases listed in Proposition cover all the possibilities for Humbert curves with conformal groups of automorphisms larger than the Humbert groups ([5, 16, 19]). For some values of k we may also obtain the groups $\mathcal{A}_4 \ltimes H$, $\mathcal{A}_5 \ltimes H$ and $\mathcal{S}_4 \ltimes H$.

2.4. Topological rigidity. We say that two pairs (S_1, H_1) and (S_2, H_2), where S_j is a closed Riemann surface and H_j a subgroup of its conformal automorphisms, are **topologically equivalent** if there is a homeomorphism $\phi : S_1 \to S_2$ so that $\phi\, H_1\, \phi^{-1} = H_2$.

Theorem 2.8: *All generalized Humbert pairs of the same type $k \geq 4$ are topologically equivalent.*

Proof. Since each Fuchsian group Γ uniformizing the quotient S/H for a generalized Humbert pair (S, H) of type $k \geq 4$ has a presentation of the form (2.4), they are all topologically equivalent by Nielsen isomorphism theorem (see [15]), and the result follows from Theorem 2.3 and the fact that the commutator subgroup Γ' is uniquely defined inside Γ. ∎

Remark 2.9: The rigidity condition given in Theorem 2.8 asserts that the $(k-2)$-complex dimensional locus of conformal classes of generalized Humbert curves of type $k \geq 4$ is connected, in the corresponding moduli space.

2.5. Fixed point free subgroups of generalized Humbert groups. In this section we study certain interesting subgroups of a generalized Humbert group acting freely on the corresponding generalized Humbert surface. More precisely, we have the following result.

Proposition 2.10: *If (S, H) is a generalized Humbert pair of type $k \geq 4$, then there are $N = k\,(k+1)/2$ subgroups $\{H_0^j\}_{j=1}^N$ of H acting freely on S such that $H_0^j \cong \mathbb{Z}_2^{k-2}$ and such that S/H_0^j is a hyperelliptic Riemann surface of genus $k-2$ for each $1 \leq j \leq N$. Moreover, if k is odd, then there is a unique subgroup H_1 of H acting freely on S such that $H_1 \cong \mathbb{Z}_2^{k-1}$ and such that S/H_1 is a hyperelliptic Riemann surface of genus $(k-1)/2$. Furthermore, $H_0^j \leq H_1$ for all $1 \leq j \leq N$.*

Proof. Consider a set of standard generators $\{h_1, \ldots, h_k\}$ for H, and let $h_{k+1} = h_1\, h_2 \ldots h_k$. Fixing any one of them, say h, consider the subgroup of H given by

$$H_0 = \langle h h_{i_1}, \ldots, h h_{i_{k-2}} \rangle,$$

where $h_{i_1}, \ldots, h_{i_{k-2}}$ are any $k-2$ different elements in $\{h_1, h_2, \ldots, h_{k+1}\} - \{h\}$. It is clear that any such H_0 has the required properties, where the hyperelliptic involution on S/H_0 is induced by h, and also that $N = k\,(k+1)/2$ is the number of such subgroups. Note that in this way we obtain all the subgroups of H isomorphic to $\mathbb{Z}_2^{k-2}$ that do not contain a standard generator, as required.

Assume now that k is odd and consider the subgroup

$$H_1 = \langle h_1h_2, h_1h_3, \ldots, h_1h_k \rangle.$$

It is not difficult to see that H_1 acts freely on S and that $R_1 = S/H_1$ is a hyperelliptic Riemann surface, with the hyperelliptic involution being induced by any h_j. For the uniqueness, assume $\mathbb{Z}_2^{k-1} \cong H_2 \leq H$ acts freely on S, and set $R_2 = S/H_2$. Again, R_2 is a hyperelliptic Riemann surface, with hyperelliptic involution induced by any h_j. Both hyperelliptic surfaces R_1 and R_2 are 2-fold branched covers of the Riemann sphere with the same conical points. It follows that the identity automorphism of S/H lifts to a conformal homeomorphism $t : R_1 \to R_2$; in particular, there exists an $h \in H$ that conjugates H_1 onto H_2. As H is Abelian, $H_1 = H_2$. ∎

Remark 2.11: If we consider $k = 5$ in Proposition 2.10, then the quotient Riemann surface $R = S/H_0$ is hyperelliptic of genus 3 and it admits a group of conformal automorphisms $U = H/H_0 \cong \mathbb{Z}_2^2$ so that R/U has signature $(0, 6; 2, \ldots, 2)$ and U contains the hyperelliptic involution. There are also non-hyperelliptic Riemann surfaces M of genus 3 admitting a group $V \cong \mathbb{Z}_2^2$ of conformal automorphisms so that M/V has signature $(0, 6; 2, .., 2)$. These non-hyperelliptic Riemann surfaces have been used in [11] to construct non-isomorphic surfaces with isomorphic Jacobians without polarization.

3. Uniqueness of generalized Humbert groups

Given a generalized Humbert curve S of type k, one may wonder about the uniqueness of the corresponding generalized Humbert group as a subgroup of Aut(S). In case $k = 4$, this follows from [16]. Our next result shows that this also holds in the case $k = 5$, by an argument using Weierstrass points, which also provides a short proof for $k = 4$, included here for the sake of completeness.

Theorem 3.1: *Generalized Humbert groups of type $k = 4$ and 5 are unique. In particular, they are normal in the total group of automorphisms of the corresponding generalized Humbert curve S. Moreover, in case $k = 4$ the set of Weierstrass points of S coincides with the set of fixed points of standard generators of the generalized Humbert group.*

Remark 3.2: The last assertion in Theorem 3.1 was already observed by Edge in [5].

Corollary 3.3: *Each Fuchsian group with presentation as in* (2.4) *and* $k \in \{4,5\}$ *is uniquely determined by its commutator subgroup.*

Corollary 3.4: *Let* S *be a genus* $g \in \{5,17\}$ *generalized Humbert curve and* $H \trianglelefteq \mathrm{Aut}(S)$ *be its generalized Humbert group. Then,* $\mathrm{Aut}(S)$ *is obtained by the lifting to* S *of* $\mathrm{Aut}_{\mathrm{orb}}(S/H)$.

3.1. Proof of Theorem 3.1. In order to prove our result, we first recall the following facts on Weierstrass points (see [6]). Let S be a closed Riemann surface of genus $g \geq 1$. If $W \subset S$ denotes the set of all the Weierstrass points in S, then

$$\sum_{p \in W} \tau(p) = g^3 - g\,,$$

where

$$1 = n_1 < n_2 < \cdots < n_g < 2g$$

is the sequence of gaps at p and $\tau(p) = \sum_{j=1}^{g}(n_j - j)$ is the weight at p.

3.1.1. *The case* $k = 4$. Since S is non-hyperelliptic, $n_2 = 2$, and since S is non-trigonal, $n_3 = 3$. Let $a \in H$ be a standard generator of H and let $p \in S$ be a fixed point of a. Consider the branched two-fold covering $P : S \to T = S/\langle a \rangle$, where T is an orbifold of signature $(1, 8; 2, \ldots, 2)$. Since there are meromorphic maps $T \to \widehat{\mathbb{C}}$ of degree 2 and 3 with only one pole at $P(p)$, by composing such meromorphic functions with P we see that $n_4 \geq 5$ and $n_5 \geq 7$. Therefore $\tau(p) \geq 3$. Since there are 40 different fixed points of standard generators of H, each one with weight at least 3, and since the sum of the weight of all Weierstrass points is 120, we obtain that $\tau(p) = 3$ and that the set of Weierstrass points in S coincides with the set of fixed points of standard generators of a generalized Humbert group. If we had two different Humbert groups acting on S, there would be a standard generator of one of them not contained in the other. As the stabilizer in $\mathrm{Aut}(S)$ of any point is cyclic, the fixed points of such a standard generator would be different from the 40 ones given by the other group, a contradiction that completes the proof in case $k = 4$.

3.1.2. *The case* $k = 5$. Let us start with the following facts.

LEMMA 3.5: *Let S be a generalized Humbert curve of type $k = 5$, H be a generalized Humbert group acting on S, and $p \in S$ be a fixed point of some standard generator h of H. Then, p is a Weierstrass point of S and*

$$\tau(p) \geq 21.$$

Proof. Consider the branched two-fold covering $P : S \to T = S/\langle h \rangle$. Then T is a generalized Humbert curve of genus 5, and the point $P(p)$ is not a fixed point of any of the standard generators of the generalized Humbert group (induced by) $H/\langle h \rangle$ acting on T. It follows from the case $k = 4$ that $P(p)$ is not a Weierstrass point of T, and hence there are meromorphic maps $f_d : T \to \widehat{\mathbb{C}}$ of degree d with only one pole, at $P(p)$, for each $d \geq 6$.

It follows that there are meromorphic maps $f_d \circ P : S \to \widehat{\mathbb{C}}$ of degree $2d$ with p as the unique pole, with d as before, and therefore

$$n_{12} \geq 13, \quad n_{13} \geq 15, \quad n_{14} \geq 17, \quad n_{15} \geq 19, \quad n_{16} \geq 21, \quad n_{17} \geq 23.$$

In this way we obtain that the minimum value that $\tau(p)$ may have is 21, as desired. ∎

Remark 3.6: We recall that for $\ell : X \to Y$ a non constant holomorphic map between compact Riemann surfaces and x any point of X, it holds that if n is a gap at $\ell(x)$, then n is a gap at x ([14]). With the notation of the above proof, we know that the gaps at $P(p)$ are $1, 2, 3, 4, 5$, and the gaps at $P(q)$ are $1, 2, 3, 5, 7$, for q a fixed point of a standard generator different from h. Therefore $1, 2, 3, 4, 5, 7$ are gaps at every fixed point of a standard generator of a generalized Humbert group of type 5.

LEMMA 3.7: *In a generalized Humbert curve of type $k = 5$, two different generalized Humbert groups of type k cannot have a common standard generator.*

Proof. Suppose that there exist two different generalized Humbert groups on a generalized Humbert curve S of type $k = 5$, say H_1 and H_2, with a common standard generator h. Then, we would have two different generalized Humbert groups on the generalized Humbert curve of type 4 given by $S/\langle h \rangle$, a contradiction. ∎

We use the notation $W_H \subset W$ to denote the Weierstrass points obtained as fixed points of the generalized Humbert group H.

Lemma 3.8: *In a generalized Humbert curve of type $k = 5$ there are at most two generalized Humbert groups.*

Proof. Let H be a generalized Humbert group on S of type $k = 5$. As each standard generator of H has exactly 16 fixed points and there are exactly 6 of them, $\#(W_H) = 96$. Now, Lemma 3.5 asserts

$$(*) \qquad \sum_{p \in W_H} \tau(p) \geq 2016 \ .$$

We also have

$$(**) \qquad \sum_{p \in W} \tau(p) = 4896.$$

As a consequence of Lemma 3.7, if there exist $N \geq 2$ different generalized Humbert groups acting on S, say $H_1, \ldots, H_N$, then $W_{H_i} \cap W_{H_j} = \emptyset$, for $i \neq j$, and the sum in $(**)$ restricted to the subset $W_{H_1} \cup \cdots \cup W_{H_N}$ will be at least $2016 \times N$. Therefore $N \in \{1, 2\}$. ∎

Now we proceed to prove the theorem for the case $k = 5$. From Lemma 3.8, we need only consider the case when there are exactly two different generalized Humbert groups on S of genus 17. Let us denote them by H_1 and H_2, respectively. For each $h \in H_2$ we set $H_h = hH_1h$. As H_h is a generalized Humbert group, $H_h \in \{H_1, H_2\}$. Since $h \in H_2$ and $H_1 \neq H_2$, necessarily $H_h = H_1$. Therefore, H_2 belongs to the normalizer of H_1 in $\mathrm{Aut}(S)$. Interchanging the roles of H_1 and H_2 we obtain that H_1 belongs to the normalizer of H_2 in $\mathrm{Aut}(S)$. As a consequence, both H_1 and H_2 are normal subgroups of $K = \langle H_1, H_2 \rangle$. Also, observe that since the orders of H_1 and H_2 are the same, both have the same index in K.

The group H_2 should induce on the orbifold S/H_1 a group of automorphisms isomorphic to some $\mathbb{Z}_2^m$ (similarly, the group H_1 should induce on the orbifold S/H_2 a group of automorphisms isomorphic to $\mathbb{Z}_2^m$ for the same value of m). As such a group is a subgroup of $PSL(2, \mathbb{C})$, we obtain that $m \in \{0, 1, 2\}$. The case $m = 0$ is not possible as $H_1 \neq H_2$.

If $m = 2$, then the six branch values of S/H_1 should be invariant under some $\mathbb{Z}_2^2$ and then, up to a Möbius transformation, they can be chosen to be 0, ∞, 1, -1, i and $-i$. Let us consider an involution $x_2 \in H_2$ inducing on S/H_1 the involution that fixes the point ∞. In this way, there is some $y \in H_1$ so that yx_2 fixes a point $q \in S$ over ∞ under the branched covering $S \to S/H_1$. Let $x_1 \in H_1$ be a standard generator that fixes the same point q. As the map yx_2 induces an involution on S/H_1, we have that $(yx_2)^2 \in \{1, x_1\}$. If $(yx_2)^2 = 1$, then yx_2 and x_1 are conformal automorphisms of order 2 on S with q as a common fixed point. As the stabilizer of a point in the group of conformal automorphisms is cyclic, it follows that $x_2 = yx_1 \in H_1$, a contradiction to the fact that x_2 is not the identity on S/H_1. If $(yx_2)^2 = x_1$, then $(yx_2y)x_2 = x_1$, which means that $x_1 \in H_2$; that is, both H_1 and H_2 share an standard generator, a contradiction to Lemma 3.7.

The remaining case is $m = 1$. In this case, every pair of standard generators a_1 and a_2 of H_2 must project to the same conformal involution on S/H_1; in particular, a_1a_2 projects to the identity and hence $a_1a_2 \in H_1$. In other words, there is a subgroup of index two inside H_2 which is also contained inside H_1, and $H_1 \cap H_2 \cong \mathbb{Z}_2^4$. This subgroup acts freely on S, as H_1 and H_2 do not have a common standard generator. It follows form the Riemann–Hurwitz formula that $R = S/H_1 \cap H_2$ is a genus 2 Riemann surface. In this situation any standard generator of H_j projects to the hyperelliptic involution on R (as the quotient of R by the projection is the Riemann sphere). Since the hyperelliptic involution is unique, it follows that if $a \in H_1$ and $b \in H_2$ are standard generators, then ab induces the identity on R; that is, $ab \in H_1 \cap H_2$. This implies that $a \in H_2$ and $b \in H_1$ and, in particular, $H_1 = H_2$, a contradiction, thus completing the proof of Theorem 3.1. ∎

Question 1: *Are generalized Humbert groups unique for $k \geq 6$?*

Question 2: *In case $k = 5$, is the set of Weierstrass points equal to the set of fixed points of the standard generators of the generalized Humbert group?*

3.2. Humbert groups are conjugated. In the previous section we have seen that in the cases $k = 4$ and $k = 5$ generalized Humbert groups are unique in the group of conformal automorphisms of the surface. For greater values of k this uniqueness is not clear. In this section we observe that any two generalized Humbert groups acting on the same Humbert curve are conjugated in the group of conformal automorphisms of the surface. We denote $H \trianglelefteq F$ for H being a normal subgroup of F.

Lemma 3.9: *Let (S, H) be a generalized Humbert pair for $k \geq 4$. If $H \trianglelefteq F \leq \mathrm{Aut}(S)$, then H is the unique generalized Humbert subgroup of type k in F. In particular, if $H \trianglelefteq \mathrm{Aut}(S)$, then H is the unique generalized Humbert subgroup of type k in $\mathrm{Aut}(S)$.*

Proof. Assume there exists $K \leq F$ with K a generalized Humbert subgroup of type k and $K \neq H$. Set $G = HK$ and $R = H \cap K$. Then $H \trianglelefteq G$, $|G : H| = |G : K|$ and $R \leq Z(G)$. The group G/H is a 2-subgroup of $\mathrm{Aut}(S/H)$, S/H has the Riemann sphere as underlying Riemann surface and $G/H \cong K/R \leq \mathbb{Z}_2^n$. As already noted earlier, then $n \leq 2$; that is, $G/H \leq (\mathbb{Z}_2)^2$ and $\exp(G) = 4$ (the exponent of the group G). We analyze the two possible cases for G/H separately.

Case (1): Suppose $|G/H| = 2$.

Consider $x \in H$ and $y \in K$ such that

$$H = R \times \langle x \rangle \quad \text{and} \quad K = R \times \langle y \rangle.$$

Hence $G = HK = R\langle x, y\rangle$ and $\langle x, y\rangle \trianglelefteq G$. Since $\exp(G) = 4$, we obtain that $\langle x, y\rangle$ is a dihedral group of order eight.

In G we have that

$$\{z_1xy, z_2yx : z_i \in R\}$$

is the set of all elements of order four and

$$\{z_1, z_2x, z_3y, z_4(xy)^2, z_5yxy, z_6xyx : z_i \in R,\ z_1 \neq 1, z_4 \neq (xy)^2\}$$

is the set of all elements of order two.

Therefore, for each set $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ of elements of G we have that

$$|\langle \alpha_1, \alpha_2, \ldots, \alpha_s\rangle| \leq 2^s\, 2^3 .$$

In particular, any generating set of G has at least $k - 2$ elements, and applying the Riemann–Hurwitz equation we see that $k \leq 7$. Therefore, we only need to analyze the cases $k = 4, 5, 6$ and 7.

Since the arguments in each case are similar, we illustrate with the case $k = 7$. By the Riemann–Hurwitz equation, the only possibility for the signature of S/G to be analyzed is $(0, 6; 2, 2, 2, 2, 2, 2)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_5$ be a generating system for G such that $\alpha_i^2 = 1,\ i = 1, \ldots, 5$ and $(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5)^2 = 1$. Since $\exp(G) = 4$, we may assume that $|\alpha_1\alpha_2| = 4$, and it follows that $U_1 = \langle \alpha_1, \alpha_2\rangle \trianglelefteq G$. If $(\alpha_3\alpha_4)^2 = 1$, $(\alpha_3\alpha_5)^2 = 1$ and $(\alpha_4\alpha_5)^2 = 1$, then $|\langle \alpha_3, \alpha_4, \alpha_5\rangle| \leq 2^3$ and $|G| = |U_1\langle \alpha_3, \alpha_4, \alpha_5\rangle| \leq 2^6$, a contradiction. So we may assume that $|\alpha_3\alpha_4| = 4$. Hence $U_2 = \langle \alpha_3, \alpha_4\rangle \trianglelefteq G$ and $|U_1 \cap U_2| \geq 2$. Therefore, $G = U_1U_2\langle \alpha_5\rangle$ and $|G| \leq 2^6$, a contradiction.

Case (2): Suppose $G/H \cong \mathbb{Z}_2^2$.

Let M be a maximal subgroup of G such that $K \leq M$. Then, by the previous case, K is the unique generalized Humbert subgroup of type k of M and since $M \trianglelefteq G$, it follows that $K \trianglelefteq G$.

Consider $u, v \in H$ and $x, y \in K$ such that $H = R \times \langle u, v\rangle$ and $K = R \times \langle x, y\rangle$. By the previous case, H is the unique generalized Humbert subgroup of type k of $H\langle x\rangle$, $H\langle y\rangle$ and $H\langle xy\rangle$. Also K is the unique generalized Humbert subgroup of type k of $K\langle u\rangle$, $K\langle v\rangle$ and $K\langle uv\rangle$. Hence $|xu| = |xv| = |yu| = |yu| = 4$, and $[x, u] = xuxu \in R$, $[x, v] \in R$, $[y, u] \in R$.

Set $U = \langle u, v, x, y\rangle$. Then $G = RU$ and $U \trianglelefteq G$. A presentation of U is given as follows.

$$U = \langle u, v, x, y \ / \ u^2, v^2, x^2, y^2, (uv)^2, (xy)^2, (xu)^4, (xv)^4,$$
$$(yu)^4, (yv)^4, (xuxv)^2, (yuyv)^2, (uxuy)^2, (vxvy)^2\rangle$$

from where $|U| = 2^8$. As in case (1), we obtain that k is odd and $k \leq 11$. The remaining particular cases $k = 5, 7, 9, 11$ are not difficult to analyze, and thus the proof is completed. ∎

Proposition 3.10: *Let (S, H) be a generalized Humbert pair for $k \geq 4$. If $K \leq \mathrm{Aut}(S)$ is a generalized Humbert subgroup of type k, then there exists $g \in \langle H, K\rangle$ such that g conjugates H to K.*

Proof. Set $F = \langle H, K\rangle$ and let $L \leq F$ be a 2-Sylow subgroup of F such that $H \leq L$. Then, by Sylow's Theorem, there exists g in F such that $K^g = gKg^{-1} \leq L$. If $H = L$, then K is also a 2-Sylow subgroup of F and the conclusion follows. Otherwise, there is a series of subgroups of L

$$H = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_r = L$$

such that $H_i \trianglelefteq H_{i+1}$, for all $i = 0, 1, \ldots, r-1$, with $r \geq 1$. Applying the previous lemma we obtain that H is unique in H_1. If $H_1 = L$, we are done. Otherwise, it follows that H is normal in H_2, and we may apply the lemma again to conclude that H is unique in H_2. Repeating the argument as needed, we obtain that H is the unique generalized Humbert subgroup of type k in L, and we conclude that $K^g = H$. ∎

4. Algebraic description

Consider the canonical holomorphic embedding

$$j_S : S \to \mathbb{CP}^{g_k - 1}$$

of a hyperbolic generalized Humbert curve S of type $k \geq 4$, and hence of genus $g_k = 1 + 2^{k-2}(k-3)$. As a consequence of the non-hyperellipticity and non-trigonality of S together with the fact that it is not a plane quintic, it follows from Petri–Noether's theorem (see [17]) that there are exactly $(g_k - 3)(g_k - 2)/2$ linearly independent quadrics through $j_S(S)$; furthermore, the quadrics can be chosen of rank at most six. In this section we provide another algebraic description by means of $(k-1)$ quadrics in $\mathbb{CP}^k$, which seems to be better suited for computations.

4.1. Algebraic curves. Let $\mathbb{CP}^k$ be the projective space with homogeneous coordinates $x_1, \ldots, x_{k+1}$ and $C(\lambda_1, \ldots, \lambda_{k-2}) \subset \mathbb{CP}^k$ the algebraic curve given by the following $(k-1)$ homogeneous polynomials of degree 2:

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 = 0 \\ \lambda_1 x_1^2 + x_2^2 + x_4^2 = 0 \\ \lambda_2 x_1^2 + x_2^2 + x_5^2 = 0 \\ \quad\vdots \qquad\qquad \vdots \quad \vdots \\ \lambda_{k-2} x_1^2 + x_2^2 + x_{k+1}^2 = 0 \end{cases}$$

where $\lambda_j \in \mathbb{C} - \{0, 1\}$, $\lambda_i \neq \lambda_j$, for $i \neq j$.

The conditions on the parameters λ_j assert that $C(\lambda_1, \ldots, \lambda_{k-2})$ is a non-singular algebraic curve; that is, a closed Riemann surface. On $C(\lambda_1, \ldots, \lambda_{k-2})$ we have the action of the Abelian group of conformal automorphisms $H_0 \cong \mathbb{Z}_2^k$ generated by the transformations

$$a_j[x_1 : \cdots : x_{k+1}] = [x_1 : \cdots : x_{j-1} : -x_j : x_{j+1} : \cdots : x_{k+1}], \quad j = 1, \ldots, k.$$

If we consider the degree 2^k holomorphic map

$$\pi : C(\lambda_1, \ldots, \lambda_{k-2}) \to \widehat{\mathbb{C}} \quad \text{given by } \pi[x_1 : \ldots : x_{k+1}] = (x_2/x_1)^2,$$

we see that $\pi a_j = \pi$, for every a_j, $j = 1, \ldots, k$. Since the branch values of π are

$$\{\infty, 0, -1, -\lambda_1, -\lambda_2, \ldots, -\lambda_{k-2}\},$$

it follows that $(C(\lambda_1, \ldots, \lambda_{k-2}), H_0)$ is a generalized Humbert pair of type k.

As every generalized Humbert pair (S, H) of type k is uniquely determined by the quotient S/H (by Theorem 2.3), the above construction implies the following result.

THEOREM 4.1: *Let (S, H) be a generalized Humbert pair of type k and let $T : S/H \to \widehat{\mathbb{C}}$ be a conformal homeomorphism so that T sends the set of conical points of S/H into the set $\{\infty, 0, -1, -\lambda_1, -\lambda_2, \ldots, -\lambda_{k-2}\}$. Then, (S, H) is conformally equivalent to $(C(\lambda_1, \ldots, \lambda_{k-2}), H_0)$.*

Remark 4.2: (i) The algebraic curve in $\mathbb{CP}^2$ given by

$$x_1^2 + x_2^2 + x_3^2 = 0$$

corresponds to the generalized Humbert curve of genus zero with generalized Humbert group $\mathbb{Z}_2^2$, and the locus in $\mathbb{CP}^3$ given by

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 = 0 \\ \lambda_1 x_1^2 + x_2^2 + x_4^2 = 0 \end{cases}$$

corresponds to the generalized Humbert curves of genus one.

(ii) The above algebraic description is well-suited to describe the parameters λ_j in order to obtain extra conformal automorphisms of S. For instance, Case (4) in Proposition 2.6 may be given by the conditions $\lambda_{2j}\lambda_{2j-1} = 1$, for $1 \leq j \leq (k-2)/2$; Case (5) by $\lambda_{2j}\lambda_{2j+1} = \lambda_1$, for $1 \leq j \leq (k-3)/2$, and Case (6) by $\lambda_1 = -1$ and $\lambda_{2j+1} = \lambda_{2j}$, for $1 \leq j \leq (k-3)/2$.

4.2. MODULI SPACES. Let us consider the set $\mathcal{H}_k$ of all generalized Humbert pairs of type k and the natural map

$$r : \mathcal{H}_k \to \mathcal{M}_{g_k} : (S, H) \mapsto [S],$$

where $\mathcal{M}_{g_k}$ is the moduli space of curves of genus $g_k = 1 + (k-3)2^{k-2}$. The image $r(\mathcal{H}_k) \subset \mathcal{M}_{g_k}$ is the locus of all conformal equivalence classes of generalized Humbert curves of genus g_k.

Let us consider the equivalence relation in $\mathcal{H}_k$ given by the conformal equivalence relation of pairs given in Section 1 and denote by $\widehat{\mathcal{H}}_k$ the set of these equivalence classes. If we denote by $p : \mathcal{H}_k \to \widehat{\mathcal{H}}_k$ the projection map, then we have a natural map $q : \widehat{\mathcal{H}}_k \to \mathcal{M}_{g_k}$ so that $r = qp$.

PROPOSITION 4.3: *The map $q : \widehat{\mathcal{H}}_k \to \mathcal{M}_{g_k}$ is injective.*

Proof. The injectivity of the map q may fail only if there is a generalized Humbert curve S admitting two non-conjugate generalized Humbert groups. In this way, the result follows from Proposition 3.10 (see also Theorem 3.1 for $k = 4$ and $k = 5$). ∎

4.3. Parameter spaces. Let us consider the parameter space

$$\mathcal{P}_k = \{(\lambda_1, \ldots, \lambda_{k-2}) \in \mathbb{C}^{k-2} : \lambda_j \neq 0, 1,\ \lambda_j \neq \lambda_s \text{ for } s \neq j\}.$$

The Möbius group $\mathrm{PSL}(2, \mathbb{C})$ acts in a natural way on the parameter set $\mathcal{P}_k$, component-wise. Two tuples in $\mathcal{P}_k$ which are equivalent under $\mathrm{PSL}(2, \mathbb{C})$ determine biholomorphic generalized Humbert curves. Therefore, the quotient

$$\mathcal{Q}_k = \mathcal{P}_k / \mathrm{PSL}(2, \mathbb{C})$$

defines a finite covering of $r(\mathcal{H}_k)$. Each generalized Humbert pair (S, H) of type k provides a tuple in $\mathcal{P}_k$ up to the above action of $\mathrm{PSL}(2, \mathbb{C})$. Proposition 3.10 (Theorem 3.1 for $k = 4, 5$) asserts that for a couple of generalized Humbert pairs (S, H) and (S, K) there is a conformal automorphism $h : S \to S$ conjugating H onto K. In this way, h induces a conformal automorphism between the orbifolds S/H and S/K. In particular, this fact together with Proposition 4.3 allows us to see the following.

Proposition 4.4:

$$\mathcal{Q}_k \cong r(\mathcal{H}_k) \cong \widehat{\mathcal{H}}_k.$$

5. Schottky uniformizations of generalized Humbert pairs

In this section we provide Schottky uniformizations for all generalized Humbert pairs (S, H). We prove that for any generalized Humbert pair (S, H) there is a Kleinian group K (called a generalized Humbert–Whittaker group) whose commutator subgroup $G = K'$ is a Schottky group that uniformizes S and such that $H = K/G$.

5.1. Schottky uniformizations. A **Schottky group of genus zero** is just the trivial group. A Schottky group of positive genus is defined as follows. Assume we have a collection of $2g$ ($g > 0$) pairwise disjoint simple loops, say C_1, C_1', …, C_g and C_g', in the Riemann sphere bounding a common region $\mathcal{D}$ of connectivity $2g$, and that there are loxodromic transformations A_1, …, A_g so that $A_j(C_j) = C_j'$ and $A_j(\mathcal{D}) \cap \mathcal{D} = \emptyset$, for each $j = 1, 2, \ldots, g$. The group

G generated by $A_1, \ldots, A_g$ is called a **Schottky group of genus** $g > 0$. The collection of loops C_1, C_1', $\ldots$, C_g and C_g', is called a **fundamental system of loops** of G respect to the generators $A_1, \ldots, A_g$.

If we denote by Ω the region of discontinuity of a Schottky group G of genus g, then the quotient $S = \Omega/G$ turns out to be a closed Riemann surface of genus g. The reciprocal is valid by the retrosection theorem [13] (see [4] for a modern proof using quasiconformal deformation theory). A triple $(\Omega, G, P : \Omega \to S)$ is called a **Schottky uniformization** of a closed Riemann surface S if G is a Schottky group with Ω as its region of discontinuity and $P : \Omega \to S$ is a holomorphic regular covering with G as covering group.

If we have a pair (S, H), where S is a closed Riemann surface and H is a group of conformal automorphisms of S, then we say that H is of **Schottky type** if there is a Schottky uniformization $(\Omega, G, P : \Omega \to S)$ of S such that, for each $h \in H$ there is a conformal automorphism $k : \Omega \to \Omega$, a Schottky lifting of h, so that $h \circ P = P \circ k$. It is known that k should then be the restriction of a Möbius transformation [2]. In this way, the group K generated by the Schottky lifting of all the automorphisms $h \in H$ turns out to be a geometrically finite function group, containing the Schottky group G as a finite index normal subgroup so that $K/G \cong H$. Necessary and sufficient conditions for a group of automorphisms of a given closed Riemann surface to be of Schottky type can be found in [8]. In the case of generalized Humbert pairs, every generalized Humbert group is of Schottky type ([9]).

A Schottky uniformization of a generalized Humbert curve S for which its generalized Humbert group H lifts will be called a **Schottky uniformization of the generalized Humbert pair** (S, H). As a consequence of the above and Theorem 2.8 we have the following.

Proposition 5.1: *Let $(\Omega_0, G_0, P_0 : \Omega_0 \to S_0)$ be a Schottky uniformization of a generalized Humbert pair (S_0, H_0). Let us denote by K_0 the geometrically finite group generated by the lifting of all automorphisms of H_0. If (S, H) is a generalized Humbert pair of the same type, then there is a quasiconformal homeomorphism $\omega : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, such that $K = \omega K_0 \omega^{-1}$ is a group of Möbius transformations and*

$$(\Omega_\omega = \omega(\Omega_0), G_\omega = \omega G_0 \omega^{-1}, P_\omega : \Omega_\omega \to S)$$

is a Schottky uniformization of the generalized Humbert pair (S, H).

As a consequence of Proposition 5.1, if we construct a geometrically finite function group K_0 which contains as a normal subgroup a Schottky group G_0 such that

$$K_0/G_0 \cong \mathbb{Z}_2^k$$

and Ω_0/K_0 has signature $(0, k+1; 2, \ldots, 2)$, then we obtain, by quasiconformal deformation of K_0, Schottky uniformizations for all generalized Humbert pairs (S, H) so that the corresponding lifted groups K are topologically (quasiconformally) conjugated. As a consequence, we need only to find one concrete example of Schottky uniformization of a generalized Humbert pair for each k.

5.2. Real generalized Humbert–Whittaker groups. In this section we provide a construction of Schottky uniformizations of generalized Humbert pairs (S, H) with the property that S admits an anticonformal involution with fixed points (a reflection) which commutes with every element of H. We consider only the case that the reflection projects to S/H as a reflection that fixes each of the conical points. Similar constructions may be done for the other possible cases.

Let us consider a chain of k circles on the complex plane, say

$$C_1, C_2, \ldots, C_k,$$

such that:

(1) C_j is orthogonal to the unit circle C_0, for $j = 1, \ldots, k$;
(2) C_j is orthogonal to C_{j+1}, for $j = 1, \ldots, k-1$;
(3) C_j is disjoint from C_i, for $i \notin \{j-1, j, j+1\}$, for $j = 2, \ldots, k-1$ and $i = 1, \ldots, k$;
(4) all of the circles $C_1, \ldots, C_k$ bound a common domain $\mathcal{D}$.

Let σ denote the reflection on the unit circle C_0, and let σ_j be the reflection on the circle C_j, for $j = 1, \ldots, k$. Consider the following elliptic transformations of order two:

$$E_j = \sigma \circ \sigma_j, \quad \text{for } j = 1, \ldots, k.$$

Then the group K generated by the transformations $E_1, \ldots, E_k$ is a geometrically finite function group (using Klein–Maskit's combination theorems [15]). The group K keeps invariant the unit circle (that is, it is an extended Fuchsian group of the second kind). Moreover, K has a presentation as follows

$$\langle E_1, \ldots, E_k : E_1^2 = \cdots = E_k^2 = (E_2E_1)^2 = (E_3E_2)^2 = \cdots = (E_kE_{k-1})^2 = 1\rangle.$$

If Ω is the region of discontinuity of K, then Ω consists of the union of the unit disc, the exterior unit disc and a countable collection of open arcs in the unit circle. The quotient orbifold Ω/K has signature $(0, k+1; 2, \ldots, 2)$. Let $G = K'$ be the commutator subgroup of K. If we consider the surjective homomorphism

$$\Phi : K \to \mathbb{Z}_2^k = \langle x_1, \ldots, x_k : x_j^2 = (x_i x_j)^2 = 1 \rangle,$$

defined by

$$\Phi(E_j) = x_j, \quad j = 1, \ldots, k,$$

then G is the kernel of Φ, and furthermore

(i) G is a torsion free normal subgroup of K;
(ii) $K/G \cong \mathbb{Z}_2^k$;
(iii) G is the smallest normal subgroup in K containing the transformations

$$(E_3E_1)^2, \ldots, (E_kE_1)^2, (E_4E_2)^2, \ldots, (E_kE_2)^2, \ldots, (E_kE_{k-2})^2,$$

The above properties assert that G is a finitely generated second kind torsion free Fuchsian group, hence a Schottky group.

Then Ω/G is a generalized Humbert curve of type k, with generalized Humbert group $H = K/G \cong \mathbb{Z}_2^k$. The group K will be called a **real generalized Humbert-Whittaker group of type** k and the corresponding Schottky group G (that is, its commutator subgroup) a **real generalized Humbert Schottky group of type** k.

Remark 5.2 (Parameters): In the above construction we have infinitely many different real generalized Humbert–Whittaker groups of the same type k. Let $K = \langle E_1, \ldots, E_k \rangle$ be a real generalized Humbert–Whittaker group of type k. Up to conjugation by a suitable Möbius transformation we may assume that the extended real line is invariant under K and that $E_1(z) = 1/z$ and $E_2(z) = (5z-4)/(4z-5)$. In this way, the group K is determined by the fixed points of $E_3, \ldots, E_k$. If we denote by $a_k < b_k$ the fixed points of E_k, then we should have $b_3 = E_2(a_3) = (5a_3-4)/(4a_3-5)$, $b_4 = E_3(a_3), \ldots, b_k = E_{k-1}(a_k)$. In particular, the parameters of K are given by the $(k-2)$ real parameters:

$$a_3, a_4, \ldots, a_k,$$

so that

$$2 < a_3 < b_3 < a_4 < b_4 < \cdots < a_k < b_k.$$

Now, each of the (normalized) real generalized Humbert–Whittaker group K has the property that the reflection $\sigma(z) = \overline{z}$ commutes with every element of it. Then the Riemann surface S uniformized by K admits a reflection induced by σ (that is, a real structure). If $H = K/G \cong \mathbb{Z}_2^k$ is the generalized Humbert group of S, then σ induces a reflection on S/H, that fixes each of the branch values. It is not hard to see (by use of quasiconformal deformation theory) that the reciprocal also holds.

THEOREM 5.3: *Let (S, H) be a generalized Humbert pair of type k so that S admits a reflection $\sigma : S \to S$, commuting with every element of H, and such that it defines a reflection on S/H that fixes each of the $(k + 1)$ conical points. Then (S, H) is uniformized by a suitable real generalized Humbert–Whittaker group.*

5.3. GENERALIZED HUMBERT–WHITTAKER GROUPS. Let K_0 be a real generalized Humbert–Whittaker group of type k. Any group K obtained by quasiconformal deformation of K_0 will be called a **generalized Humbert–Whittaker group of type** k. The commutator subgroup $G = K'$ turns out to be a Schottky group (obtained as the image of G_0 by such a deformation) and we call it a **generalized Humbert Schottky group of type** k. The corresponding quasicircles obtained as images of the circles $C_1, \dots, C_k$ will be called a **fundamental set of loops** of K with respect to the generators $E_1, \dots, E_k$.

As a consequence of Proposition 5.1 and the above constructions, we obtain the following result.

THEOREM 5.4: *Let (S, H) be a generalized Humbert curve of type k. Then, there exists a generalized Humbert–Whittaker group K of type k, with corresponding generalized Humbert Schottky group $G = K'$ of type k (its commutator subgroup), such that S is uniformized by G and S/H is uniformized by K.*

Remark 5.5: Consider a hyperbolic generalized Humbert pair (S, H) of type $k \geq 4$. We know that S can be uniformized by the commutator subgroup Γ' of a Fuchsian group Γ uniformizing S/H so that $H = \Gamma/\Gamma'$. The group Γ can be constructed from a hyperbolic polygon as follows. Consider a compact hyperbolic polygon $P \subset \mathbb{H}^2$ with k sides and such that the sum of all internal angles is π. The sides of P are denoted by $\sigma_1, \dots, \sigma_k$, in counterclockwise order. Let x_j be the order two elliptic isometry whose fixed point is the middle point

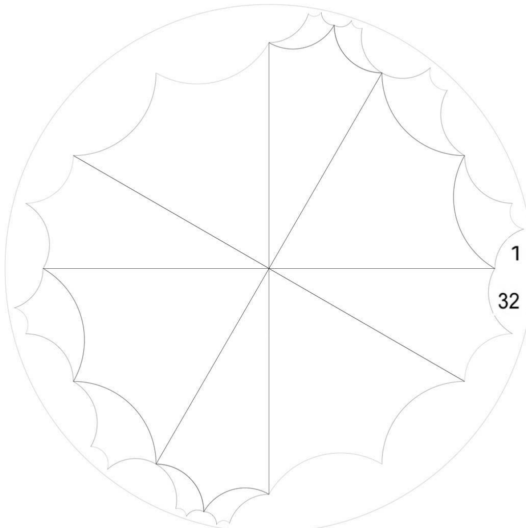

Figure 1. Fundamental domains for Γ' and Γ when $k = 4$

of the side σ_j. As a consequence of Poincaré's hyperbolic polygon theorem, the group $\widehat{\Gamma}$ generated by the involutions $x_1, \ldots, x_k$ is a Fuchsian group with P as fundamental domain. Moreover, a complete set of relations for $\widehat{\Gamma}$ is given by

$$x_1^2 = \cdots = x_k^2 = (x_1 x_2 \cdots x_k)^2 = 1$$

and $\mathbb{H}^2/\widehat{\Gamma}$ has signature $(0, k+1; 2, \ldots, 2)$. As a consequence of quasiconformal deformation theory, we may assume that $\widehat{\Gamma} = \Gamma$. A fundamental region for Γ' is obtained by an appropriate gluing of 2^k copies of P; Figure 1 illustrates the case $k = 4$, where the external sides are numbered anticlockwise from 1 to 32, and the pairing is given by

$$1-24,\ 2-11,\ 3-30,\ 4-13,\ 5-28,\ 6-23,\ 7-22,\ 8-17,$$
$$9-32,\ 10-15,\ 12-21,\ 14-19,\ 16-25,\ 18-27,\ 20-29,\ 26-31.$$

Consider N, the smallest normal subgroup of Γ containing the elements

$$(x_2x_1)^2, (x_3x_2)^2, \ldots, (x_{k+1}x_k)^2,$$

where $x_{k+1} = x_1 x_2 \ldots x_k$. Since the axes of any two of these hyperbolic transformations do not intersect, we have that $\Omega = \mathbb{H}^2/N$ is a planar surface. The group $K = \Gamma/N$ is a group of automorphisms of Ω, and it is a generalized

Humbert–Whittaker group of type k that uniformizes S/H. Its commutator subgroup $G = K'$ (that is, its generalized Humbert–Schottky group of type k) is a Schottky group that uniformizes S.

5.4. Extended generalized Humbert–Whittaker groups. In this section we construct a certain class of groups which contains a generalized Humbert–Whittaker group as a normal subgroup of index two, and such that the corresponding generalized Humbert–Schottky group is also normal. In particular, the generalized Humbert curve uniformized by such a generalized Humbert–Schottky group admits a $\mathbb{Z}_2$-extension of a generalized Humbert group as a group of conformal automorphisms. We first do the construction for the real case.

5.4.1. *Real extended generalized Humbert–Whittaker groups.* Let us consider a chain of $l+1$ circles on the complex plane, say

$$C_1, C_2, \ldots, C_l, D$$

so that:

(1) D and C_j are orthogonal to the unit circle C_0, for $j = 1, \ldots, l$;
(2) C_j is orthogonal to C_{j+1}, for $j = 1, \ldots, l-1$;
(3) C_j is disjoint from C_i, for $i \notin \{j-1, j, j+1\}$, for $j = 2, \ldots, l-1$ and $i = 1, \ldots, l$;
(4) D is disjoint from all circle C_j, for $j = 1, \ldots, l-1$;
(5) D intersects C_l at exactly two points, and the intersection angle is π/m, where $m \in \{2, 4\}$;
(6) all of the circles $C_1, \ldots, C_l$, D bound a common domain $\mathcal{D}$.

Let σ be, as before, the reflection on the unit circle C_0, σ_j be the reflection on the circle C_j, for $j = 1, .., l$, and σ_D be the reflection on D. Define the following elliptic transformations of order two:

$$E_j = \sigma\sigma_j, \quad \text{for } j = 1, \ldots, l,$$

$$F = \sigma\sigma_D.$$

The group $\widehat{K}_0$ generated by the transformations $E_1, \ldots, E_l$ and F is a geometrically finite function group (using Klein–Maskit's combination theorems [15]), which keeps invariant the unit circle (that is, it is an extended Fuchsian group of the second kind). If Ω is the region of discontinuity of $\widehat{K}_0$, we have that $\Omega/\widehat{K}_0$ has signature $(0,4;2,2,2,4)$. Moreover, the group $\widehat{K}_0$ has presentation

$$\widehat{K}_0 = \langle E_1, \dots, E_l, F : E_1^2 = \cdots = E_l^2 = F^2 = 1$$

$$(E_2E_1)^2 = (E_3E_2)^2 = \cdots = (E_lE_{l-1})^2 = (FE_l)^m = 1\rangle, \quad m \in \{2,4\}.$$

The group $\widehat{K}_0$ will be called a **real extended generalized Humbert–Whittaker group**.

Theorem 5.6: *The real extended generalized Humbert–Whittaker group $\widehat{K}_0$ contains a real generalized Humbert–Whittaker group as a normal subgroup of index two. In particular, the corresponding generalized Humbert Schottky group is also a normal subgroup, then of index four, of $\widehat{K}_0$.*

Proof.

1.-. In the case $m=4$, we consider

$$E_{l+r} = F \circ E_{l+1-r} \circ F, \quad r = 1, \dots, l.$$

The group K_0 generated by $E_1, \dots, E_{2l}$ turns out to be a real generalized Humbert–Whittaker group. The only relation that needs to be verified is $(E_{l+1} \circ E_l)^2 = 1$, which is a consequence of the relation $(FE_l)^4 = 1$.

2.-. In the case $m=2$, we consider

$$E_{l+r} = FE_{l-r}F, \quad r = 1, \dots, l-1.$$

The group K_0 generated by $E_1, \dots, E_{2l-1}$ turns out to be a real generalized Humbert–Whittaker group.

3.- *Normality of the generalized Humbert–Schottky group.* Let G_0 be the real generalized Humbert–Schottky group of K_0 above. Since G_0 is the commutator subgroup of K_0, we clearly have $G_0 \lhd \widehat{K}_0$. ∎

5.4.2. *Extended generalized Humbert–Whittaker groups.* Let us consider a real extended generalized Humbert–Whittaker group $\widehat{K}_0$. Any group $\widehat{K}$ obtained by quasiconformal deformation of K_0 will be called an **extended generalized Humbert–Whittaker group**. As a consequence of the above construction, we have uniformized the families described in Cases (4) and (5) of Proposition 2.6.

THEOREM 5.7: *Let $\widehat{K}$ be an extended generalized Humbert–Whittaker group of type k and let K be its index two generalized Humbert group with respective generalized Humbert–Schottky group $G = K' \trianglelefteq \widehat{K}$. Then the generalized Humbert curve S uniformized by G admits a group $\widehat{H} = \widehat{K}/G$ of conformal automorphisms isomorphic to a $\mathbb{Z}_2$-extension of the generalized Humbert group H; the corresponding generalized Humbert pair (S, H) belongs to one of the families defined in case (4) or (5) of Proposition 2.6, depending on the parity of k. Conversely, every such pair may be so obtained.*

Remark 5.8: The cases (1), (2), (3) and (6) of Proposition 2.6 cannot be described in terms of Schottky uniformizations, as they do not satisfy the necessary conditions given in [8].

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